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Euclidean Markov fields of higher integer spin I. Massive case

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Abstract. The general Lagrangian approach of Takahashi and Umezawa is used for the construction of free massive spin-1 and spin-2 Euclidean Markov fields. It is shown that these fields can be put in the framework of Markov field theory along the lines of Nelson. An example of a Euclidean field which is Markovian but non-reflexive is also given. This field does not lead to a Wightman theory in the Minkowski region.

1. Introduction

Euclidean quantum fields of arbitrary spin have been studied by Ozkaynak (1974) in the spirit of the Osterwalder and Schrader axioms (Osterwalder and Schrader 1973, 1975). However, he has not discussed the Markov property or reflection property of these fields. Gross has constructed a Euclidean Proca field (Gross 1975), but his remark that such a field is non-Markovian is incorrect. A proof of the Markov property for a Euclidean Proca field was given by Yao (1975). A Euclidean massive spin-1 field in terms of a rank-two antisymmetric tensor has also been shown to be Markovian by Lim (1975). It is also possible to formulate a Euclidean vector meson field in covariant R_ξ gauges, which is Markovian but non-reflexive and does not lead to a Wightman theory (Lim 1975). In this paper we shall show that there exists a general method of constructing Euclidean Markov fields for massive particles with spin $S \leqslant 2$.

Nelson's proof of the Markov property for Euclidean scalar field depends crucially on the fact that the inverse of the Euclidean propagator exists and is a local differential operator (Nelson 1973a, b). This is closely related to the Lagrangian field theory expounded by Takahashi and Umezawa (Umezawa and Takahashi 1953, Takahashi 1969). The main idea is as follows. A field of massive particles with higher spin, described by a tensor, has too many components to describe particles of a unique spin. Some of the lower spins enter with negative metrics in the Wightman functions, and have to be eliminated by imposing subsidiary conditions on the field. The method proposed by Takahashi and Umezawa is to express the field equation in the form of a single local differential matrix equation

$$\Lambda(\partial)\phi(x) = 0 \tag{1.1}$$

such that it can be reduced to the Klein–Gordon equation and all the subsidiary conditions by a finite number of differentiations and algebraic operations. In other words there exists a differential operator $d(\partial)$ called the Klein–Gordon divisor, which satisfies

$$d(\partial)\Lambda(\partial) = \Lambda(\partial)\, d(\partial) = (\Box + m^2)I. \tag{1.2}$$

Furthermore, there is a non-singular matrix η defined by

$$[\eta\Lambda(\partial)]^* = \eta\Lambda(-\partial) \tag{1.3}$$

so that the equation of motion (1.1) can be derived by variational method from the following local Lagrangian density

$$\mathscr{L} = -\phi^*(x)\eta\Lambda(\partial)\phi(x) \tag{1.4}$$

where the asterisk denotes Hermitian conjugate.

The free propagator is just the matrix inverse of $\Lambda(\partial)$, i.e. $d(\partial)(\Box+m^2)^{-1}$. The locality of $\Lambda(\partial)$ ensures the Markov property of the corresponding Euclidean field. Thus, the Markov property is closely related to the possibility of finding a local Lagrangian density for the field.

2. Euclidean massive spin-1 fields

We shall study three models, namely the Euclidean Proca field, the antisymmetric rank-two tensor field and the vector meson field in covariant R_ξ gauges.

2.1. *Euclidean Proca field*

The relativistic equation of motion for particles with mass m and spin equal to 1 is the Proca equation

$$\Lambda_{\mu\nu}(\partial)\phi^\nu(x) = 0 \tag{2.1a}$$

where

$$\Lambda_{\mu\nu}(\partial) = -(\Box+m^2)g_{\mu\nu} + \partial_\mu\partial_\nu. \tag{2.1b}$$

Here we have used the convention $g_{00} = +1$, $g_{ij} = -\delta_{ij}$ for $i = 1, 2, 3$. The Klein–Gordon divisor is given by

$$d^{\mu\nu} = -(g^{\mu\nu} + m^{-2}\partial^\mu\partial^\nu) \tag{2.2}$$

satisfying

$$d^{\mu\nu}(\partial)\Lambda_{\nu\mu'}(\partial) = \delta^\mu_{\mu'}(\Box+m^2). \tag{2.3}$$

The two-point Wightman function is

$$W^{\mu\nu}(x-y) = \langle\Omega_0\phi^\mu(x)\phi^\nu(y)\Omega_0\rangle$$

$$= (2\pi)^{-3}\int\frac{d^3p}{2\omega(\boldsymbol{p})}\,e^{-i\omega(\boldsymbol{p})(x_0-y_0)}\,e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}(-g^{\mu\nu}+m^{-2}p^\mu p^\nu) \tag{2.4}$$

where $\omega(\boldsymbol{p}) = (\boldsymbol{p}^2+m^2)^{1/2}$. The relativistic one-particle space $\mathscr{M}$ is defined as the completion of the inner product space whose elements are equivalence classes of elements of $\mathscr{S}^4(\mathbb{R}^4) \equiv \mathscr{S}(\mathbb{R}^4)\times\mathscr{S}(\mathbb{R}^4)\times\mathscr{S}(\mathbb{R}^4)\times\mathscr{S}(\mathbb{R}^4)$, with equivalence defined with respect to the norm given by the inner product

$$\langle f, g\rangle_{\mathscr{M}} = \tfrac{1}{2}\sum_{\mu,\nu}\iint dx\,dy\,f_\mu(x)W^{\mu\nu}(x-y)g_\nu(y). \tag{2.5}$$

Since the support of the Fourier transform of $W^{\mu\nu}$ is confined to the hyperboloid sheet

$p^2=m^2$, $p_0\geqslant 0$; and $\Sigma_\mu \partial_\mu \phi^\mu = 0$ (or $\Sigma_\mu \partial_\mu W^{\mu\nu}=0$) each equivalence class may then be represented by an element $f(\boldsymbol{x}) \in \mathscr{S}^3(\mathbb{R}^3)$; with this representation, $\mathscr{M}$ has the following norm:

$$\|f\|^2_{\mathscr{M}} = \|f\|^2_{-1/2} + m^{-2}\|\text{div}\, f\|^2_{-1/2} < \infty \tag{2.6}$$

where

$$\text{div}\, f = \sum_{i=1}^{3} \partial_i f_i(\boldsymbol{x})$$

and $\|\ldots\|_{-1/2}$ is the $\mathscr{H}_{-1/2}$ Sobolev norm defined by $\|f\|_{-1/2}(-\Delta+m^2)^{-1/2}f\|_{\mathscr{L}^2}$. The physical Hilbert space for the free Proca field is the Fock space $\mathscr{F}(\mathscr{M})$.

Now the transition from the relativistic propagator to the Euclidean propagator is not as direct as that in the scalar case. The main difference lies in the fact that the Minkowski metric $g_{\mu\nu}$ appears in the propagator and no amount of analytic continuation is going to change the indefinite $g_{\mu\nu}$ into the definite δ_{ij} needed for a probabilistic interpretation. One can overcome this difficulty by re-introducing the 'old fashioned' four vector $\phi_i(x)$, $i=1,2,3,4$, with $\phi_4(x)=\mathrm{i}\phi_0(x)$. It is the Schwinger functions of this four vector field that are covariant under real Euclidean groups. Actually such a step is equivalent to the following matrix transformation

$$\phi_i(x) = A_{i\mu}\phi^\mu(x) \tag{2.7}$$

where

$$A_{i\mu}=1 \qquad \text{if } i=\mu=1,2,3,$$

$$A_{40}=\mathrm{i} \qquad \text{and } A_{i\mu}=0 \text{ otherwise.}$$

Therefore the two-point Schwinger function is given by

$$S_{ij}(x-y) = A_{i\mu}A_{j\nu}W^{\mu\nu}(\boldsymbol{x}-\boldsymbol{y}, \mathrm{i}(x_0-y_0)) = (\delta_{ij} - m^{-2}\partial_i\partial_j)S(x-y) \tag{2.8}$$

where $S(x-y)$ is the two-point Schwinger function for the massive scalar field. Its Fourier transform has a local inverse $(p^2+m^2)\delta_{ij} - m^{-2}p_ip_j$, which ensures the Markov property for the Euclidean Proca field.

The Euclidean one-particle space $\mathscr{K}$ can then be defined as the completion of the vector-valued real test-function space $\mathscr{S}^4(\mathbb{R}^4)$ with respect to the inner product

$$\langle f, g\rangle_{\mathscr{K}} = \sum_{i,j=1}^{4} \iint f_i(x)S_{ij}(x-y)g_j(y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.9a}$$

and such that

$$\|f\|^2_{\mathscr{K}} = \|f\|^2_{-1} + m^{-2}\,\|\text{div}\, f\|^2_{-1} < \infty \tag{2.9b}$$

where

$$\text{div}\, f = \sum_{i=1}^{4} \partial_i f_i(x)$$

and $\|\ldots\|_{-1}$ is a $\mathscr{H}_{-1}$ Sobolev norm. Gross (1975) has shown that $\mathscr{K}$ can be obtained by dilation of the semigroup $\mathrm{e}^{-h_0 t}$, where h_0 is the Hamiltonian for a single free particle of mass m and spin 1. Our remarks show that this dilation is connected to analytic continuation, just as it is in Nelson's theory of spin 0.

Euclidean Proca field Φ is defined as the generalized random vector field indexed by $\mathscr{K}$ with mean zero and covariance $E[\Phi(f)\Phi(g)] = \langle f, g\rangle_{\mathscr{K}}$. The Euclidean invariance of the inner product in $\mathscr{K}$ allows the full Euclidean group ISO(4) on the underlying probability space (Ω, Σ, μ) of Φ to be presented by measure-preserving automorphisms of the σ algebra Σ. The translational subgroup acts ergodically on Σ if Σ is a minimal σ algebra.

Φ does not satisfy the subsidiary conditions $\Sigma_i \partial_i \Phi_i = 0$. To see what happens to the subsidiary conditions we introduce the following definition.

Definition 1. A Euclidean field is ultralocal if all its cumulants $E_T[\Phi(x_1)\ldots\Phi(x_n)]$, $n = 2, 3, \ldots$, (i.e. truncated expectation values) are zero, unless all $x_1, \ldots, x_n$ are equal. If we assume that the first moment vanishes, then the Wightman field obtained from an ultralocal Euclidean field is zero. This is because, by definition, the Wightman functions are obtained by analytic continuation of the Euclidean Green functions evaluated at unequal points, at which points they vanish. The Euclidean Proca field does not satisfy $\Sigma_i \partial_i \Phi_i(x) = 0$ even though its Wightman field $\phi^\mu(x)$ satisfies $\Sigma_\mu \partial_\mu \phi^\mu(x) = 0$. However, $\Sigma_i \partial_i \Phi_i(x)$ is ultralocal. Indeed

$$\tilde{E}\left[\sum_i \partial_i \Phi_i \sum_j \partial_j \Phi_j\right] = \sum_{i,j} \frac{p_i p_j}{p^2 + m^2}\left(\delta_{ij} + \frac{p_i p_j}{m^2}\right) = \frac{p^2}{m^2}. \tag{2.10}$$

Then, the covariance function in x space is $-(\Delta/m^2)\delta^4(x-y)$ leading to an ultralocal field. Quantum fields with this property are related to infinitely divisible group representation (Streater 1969, 1971).

One can also show that the Euclidean Proca field satisfies the reflection property of Nelson, which can be generalized to arbitrary tensor fields as follows (Lim 1975, Yao 1976). Let τ be a representation of full Euclidean group ISO(4) on the underlying probability space (Ω, Σ, μ) such that

$$\tau(a, R)\Phi_{i_1\ldots i_s}(f) = \sum_{j_1\ldots j_s} R^{-1}_{i_1 j_1}\ldots R^{-1}_{i_s j_s}\Phi_{j_1\ldots j_s}(f_{a,R}) \tag{2.11}$$

where $a \in \mathbb{R}^4$ and $R \in \mathrm{SO}(4)$ and $f_{a,R} = f(R^{-1}(x-a))$. If $\tau(\rho)$ is the reflection in the hyperplane $x_4 = 0$ (denoted by π_0), then

$$[\tau(\rho)\Phi(f)]_{i_1\ldots i_s} = (-1)^{\sum_{r=1}^{s}\delta_{i_r 4}}\Phi_{i_1\ldots i_s}(f_l) \tag{2.12}$$

where $f_l(x) = f(\boldsymbol{x}, -x_4)$.

Definition 2. The Euclidean tensor field $\Phi_{i_1\ldots i_s}(f)$ is said to satisfy the reflection property if $\tau(\rho)u = u$, where $u \in \mathscr{L}^2(\Omega, \Sigma_0, \mu)$, Σ_0 is the sub-σ algebra generated by $\{\Phi_{i_1\ldots i_s}(f) | f \in \mathscr{K}, \sup f \subset \pi_0\}$.

We can summarize the main results for Euclidean Proca fields in the following theorems.

Theorem 1 (*Gross* 1975). The time-zero Euclidean one-particle subspace $\mathscr{K}_0$ is naturally identical to the relativistic one-particle space $\mathscr{M}$.

Theorem 2 (*Gross* 1975). If J_t is an isometric embedding of $\mathscr{F}(\mathscr{M})$ into $\mathscr{L}^2(\Omega, \Sigma, \mu)$ with image $\mathscr{L}^2(\Omega, \Sigma_t, \mu)$, then the free Hamiltonian H_0 in $\mathscr{F}(\mathscr{M})$ is related to J_t by the Feynman–Kac formula:

$$\mathrm{e}^{-|t-s|H_0} = J_s^* J_t.$$

Theorem 3 (*Yao* 1975). The Euclidean Proca field is Markovian.

Theorem 4 (*Lim* 1975, *Yao* 1976). The Euclidean Proca field satisfies the reflection property.

2.2. *Euclidean spin-1 massive tensor field*

From the group-theoretical point of view, there is an alternative way of describing massive spin-1 particles, namely, by an antisymmetric rank-two tensor $\psi^{\mu\nu}(x)$ satisfying the following equations:

$$(\Box + m^2)\psi^{\mu\nu}(x) = 0 \tag{2.13}$$

$$\psi^{\mu\nu}(x) + \psi^{\nu\mu}(x) = 0 \tag{2.14}$$

and

$$\partial_\mu \psi^{\mu\nu}(x) = 0 \tag{2.15}$$

Using the method of Takahashi and Umezawa it is possible to combine these equations into one single matrix equation:

$$\Lambda_{\mu\nu\rho\sigma}(\partial)\psi^{\rho\sigma}(x) = 0, \tag{2.16a}$$

with

$$\Lambda_{\mu\nu\rho\sigma}(\partial) = \tfrac{1}{2}(\Box + m^2)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) - \tfrac{1}{2}(g_{\mu\rho}\partial_\nu\partial_\sigma - g_{\mu\sigma}\partial_\nu\partial_\rho + g_{\nu\sigma}\partial_\mu\partial_\rho - g_{\nu\rho}\partial_\mu\partial_\sigma) - \tfrac{1}{2}\lambda m^2(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) \tag{2.16b}$$

$\lambda \neq 0$ is any real constant. This equation can be derived by the variation principle from the following local Lagrangian density:

$$\mathscr{L} = \psi^*_{\mu\nu}(x) g^{\mu\kappa} g^{\nu\tau} \Lambda_{\kappa\tau\rho\sigma}(\partial)\psi^{\rho\sigma}(x). \tag{2.17}$$

The relativistic Green function is $\mathrm{d}^{\mu\nu\rho\sigma}(\partial)(\Box + m^2)^{-1}$, with the Klein–Gordon divisor given by

$$\mathrm{d}^{\mu\nu\rho\sigma}(\partial) = \tfrac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + \frac{1}{2m^2}(g^{\nu\rho}\partial^\mu\partial^\rho - g^{\nu\rho}\partial^\mu\partial^\sigma + g^{\mu\sigma}\partial^\nu\partial^\rho - g^{\mu\rho}\partial^\nu\partial^\sigma) - \frac{\Box + m^2}{2\lambda m^2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{2.18}$$

If the field is assume *a priori* antisymmetric, then the symmetric terms in both equations (2.16) and (2.18) vanish. However, this does not alter the Wightman theory since the symmetric dropped in $\mathrm{d}(\partial)$ is proportional to $(\Box + m^2)$, and $(\Box + m^2)W(x-y) = 0$, where $W(x-y)$ is the scalar two point Wightman function. The Green function may be generalized to a one-parameter family $\mathrm{d}^{\mu\nu\rho\sigma}(\alpha, \partial)(\Box + m^2)^{-1}$, where

$$\mathrm{d}^{\mu\nu\rho\sigma}(\alpha, \partial) = \tfrac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + \frac{\alpha}{2m^2}(g^{\nu\sigma}\partial^\mu\partial^\rho - g^{\nu\rho}\partial^\mu\partial^\sigma + g^{\mu\sigma}\partial^\nu\partial^\rho - g^{\mu\rho}\partial^\nu\partial^\sigma), \tag{2.19}$$

and α is any real number. Now the inverse of $\mathrm{d}^{\mu\nu\rho\sigma}(\alpha, \partial)$ becomes

$$(\Box + m^2)(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \frac{1}{2}\left(1 + (\alpha - 1)\frac{m^2}{m^2 - \alpha\Box}\right) \times (g^{\mu\rho}\partial^\nu\partial^\sigma - g^{\mu\sigma}\partial^\nu\partial^\rho + g^{\nu\rho}\partial^\mu\partial^\sigma - g^{\nu\sigma}\partial^\mu\partial^\rho),$$

which is non-local except for $\alpha = 1$. Therefore only the case $\alpha = 1$ is considered for the construction of Markov field. Then the two-point Wightman function is

$$W^{\mu\nu\rho\sigma}(x-y) = d^{\mu\nu\rho\sigma}(\partial)W(x-y). \tag{2.20}$$

The relativistic one-particle space $\mathcal{M}$ can be defined as the completion of the space of real antisymmetric tensor test function space $\mathcal{S}^6(\mathbb{R}^3)$ with respect to the inner product,

$$\langle f, g\rangle_{\mathcal{M}} = \Sigma \int\int f_{\mu\nu}(x) W^{\mu\nu\rho\sigma}(x-y) g_{\rho\sigma}(y)\, dx\, dy \tag{2.21}$$

where $f_{\mu\nu}(x)$, $g_{\rho\sigma}(y) \in \mathcal{S}(\mathbb{R}^3)$, and $f_{\mu\nu} + f_{\nu\mu} = 0$, and such that the norm

$$\|f\|^2_{\mathcal{M}} = 2[\|f\|^2_{-1/2} + 2m^{-2}\|\operatorname{div} f\|^2_{1/2}] < \infty \tag{2.22}$$

where $\operatorname{div} f = \Sigma^3_{\mu=1} \partial^\mu f_{\mu\nu}$.

In order to obtain the correct Schwinger function one can either continue to pure imaginary time the Wightman function of the tensor fields ψ^{ij} with $\psi^{4j} = i\psi^{0j}$; or one can use the following matrix transformation:

$$\begin{aligned} S_{ijmn}(x-y) &= A_{i\mu}A_{j\nu}A_{m\rho}A_{n\sigma}W^{\mu\nu\rho\sigma}(\boldsymbol{x}-\boldsymbol{y}, i(x_0-y_0)) \\ &= [(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) + 2m^{-2}(\delta_{im}\partial_j\partial_n - \delta_{in}\partial_j\partial_m + \delta_{jm}\partial_i\partial_n - \delta_{jn}\partial_i\partial_m)]S(x-y). \end{aligned} \tag{2.23}$$

S_{ijmn} is positive definite in the antisymmetric subspace and one can construct the Euclidean one-particle space $\mathcal{K}$ in the usual manner. The norm in $\mathcal{K}$ is given by

$$\|f\|^2_{\mathcal{K}} = 2[\|f\|^2_{-1} + m^{-2}\|\operatorname{div} f\|^2_{-1}] < \infty, \tag{2.24}$$

with $\operatorname{div} f = \Sigma^4_{i=1} \partial_i f_i(x)$ and $f_{ij} + f_{ji} = 0$, $f_{ij} \in \mathcal{S}(\mathbb{R}^4)$. This norm is clearly translational invariant. If $R \in SO(4)$, then $\operatorname{div}[Rf(R^{-1}x)] = \operatorname{div} f(R^{-1}x)$. This, together with the fact that the Sobolev norm $\|\ldots\|_{-1}$ is invariant under SO(4), enables one to conclude that the induced action on $\mathcal{K}$ by ISO(4) is unitary. For a unitary representation τ of ISO(4).

$$\tau(a, R) f_{ij}(x) \tau^{-1}(a, R) = R_{ii'}R_{jj'}f_{i'j'}(R^{-1}(x-a)). \tag{2.25}$$

Now we can define a Euclidean tensor field Ψ as the generalized Gaussian random tensor field over $\mathcal{K}$ with mean zero and covariance given by $E[\Psi(f)\Psi(g)] = \langle f, g\rangle_{\mathcal{K}}$. The analogues of theorems 1–4 hold for the Euclidean tensor field Ψ. The proofs are quite similar (Lim 1975) so we shall omit them except by noting that the inverse of S_{ijmn} is a local differential operator

$$(S_{ijmn})^{-1} = -\tfrac{1}{2}(-\Delta + m^2)(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) - \tfrac{1}{2}\delta_{jn}\partial_i\partial_m - \delta_{jm}\partial_i\partial_n + \delta_{im}\partial_j\partial_n - \delta_{in}\partial_j\partial_m) \tag{2.26}$$

which guarantees the Markov property of Ψ; and the reflection property holds because

$$\tau(\rho)\Psi_{ij}\tau^{-1}(\rho) = (-1)^{\delta_{i4}+\delta_{j4}}\Psi_{ij}(f_\rho) = \Psi_{ij}(f) \tag{2.27}$$

for all $f \in \mathcal{K}_0$. Thus again we have an example of spin-1 massive Euclidean tensor field which exactly fits into the probabilistic framework of Nelson.

2.3. *Euclidean vector meson field in R_ξ gauges*

Now we shall consider a theory of vector mesons which differs considerably from those in §§ 2.1 and 2.2. In the renormalizable theory of vector mesons (Fradkin and Tyutin

1974) it is usual to employ the following free propagator (in momentum space):

$$D^{\mu\nu}(p) = -\left(g^{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi m^2}\right)\frac{1}{p^2 + m^2} \tag{2.28}$$

with one parameter family of covariant R_ξ gauges, where ξ is a real positive number. This propagator falls off like p^2 as in the scalar theory. Thus one has achieved renormalizability by the method of regularization, at the expense of introducing unphysical (or ghost) states. In the relativistic theory, the Gupta–Bleuler formalism using indefinite metric Hilbert space has to be used. The scalar ghosts have to be eliminated by imposing a subsidiary condition on the physical states.

It is interesting to note that in the Euclidean region we get a positive definite two-point Schwinger function (in momentum space)

$$S_{ij}(p) = (p^2 + m^2)^{-1}[\delta_{ij} - (1 - \xi(p^2)p_i p_j(p^2 + \xi(p^2)m^2)^{-1}] \tag{2.29}$$

which is obtained by matrix transformation $A_{i\mu}A_{j\nu}S_{ij}(p)$, and the generalization of ξ to $\xi(p^2)$, a positive measurable function. In the limit $\xi \to \infty$ we get the Euclidean propagator for the Proca field. One can now define a Euclidean vector field θ in the usual manner with mean zero and covariance in terms of S_{ij}. θ satisfies the following theorems.

Theorem 5. θ is Markovian provided $\xi^{-1}(p^2)$ is a polynomial in p^2.

Proof. Note that S_{ij} has an inverse

$$S_{ij}^{-1} = (p^2 + m^2)\delta_{ij} + (\xi^{-1}(p^2) - 1)p_i p_j,$$

which is local if $\xi^{-1}(p^2)$ is a polynomial in p^2. Then the rest of the proof follows from Nelson's argument.

We remark that even though S_{ij} with $\xi = 0$ (corresponding to Landau or transverse gauge) is positive semi-definite, it is now singular and cannot be inverted. Even if we restrict the physical space to the subspace of distributions satisfying $\Sigma_i\, \partial_i f_i(x) = 0$, one still cannot show that it is Markovian in the sense of Nelson. This differs from the Euclidean electromagnetic potential, which is Markovian in Landau gauge (Lim 1975 or the following paper).

Theorem 6. θ does not satisfy the reflection property.

Proof. The proof is simple. We just need to show that the reflection property does not hold for a certain class of test functions. The 4–4 component of the Euclidean propagator contains the term $p_4^2[(p^2 + m^2)(p^2 + \xi(p^2)m^2]^{-1}$ which allows test functions localized at the hyperplane $x_4 = 0$ of the form $f_i(\boldsymbol{x}) \otimes \delta(x_4)$ with $f_4 \neq 0$ and $f_i \in \mathscr{S}(\mathbb{R}^3)$. For such a test function we have

$$\tau(\rho)f_4(x) = -f_4(x)$$

Therefore

$$\tau(\rho)\theta(f) \neq \theta(f).$$

In the present case the Euclidean propagator is less singular than that of the Euclidean Proca field which contains 4–4 components $p_4^2 m^{-2}(p^2+m^2)^{-1}$, thus ruling out test functions localized at the hyperplane $x_4=0$, of the form $f_4 \otimes \delta(x_2)$, and ensuring the reflection property. The effect of the reflection property may be considered as to prevent the theory from being too regular in its ultraviolet behaviour. As can be seen in Nelson's theory, the reflection property excludes scalar boson fields with covariance functions such as $(-\Delta+m^2)^n, n>1$, which are regularized propagators without ultraviolet divergences. Actually such fields give rise to indefinite metric Hilbert spaces with ghost states (or non-local theories without ghost states), hence do not form Wightman theories. Furthermore, the failure of the Euclidean vector field θ to satisfy the reflection property implies that the free Hamiltonian is not self-adjoint and the analogues of theorem 3 and theorem 4 do not hold in this case. However, self-adjointness of the Hamiltonian can be achieved if we restrict to physical Hilbert space with positive metric. Therefore we conclude that the Markov property (assuming other conditions of Nelson are satisfied) is not enough to guarantee that a Euclidean field will lead to a Wightman field; the relection property must also be satisfied.

3. Euclidean massive spin-2 field

The Euclidean Markov tensor field for massive spin-2 particles can be constructed similarly to the spin-1 case. In the relativistic theory, a massive spin-2 particle can be described by a rank-two tensor field $\phi^{\mu\nu}$ which satisfies the following equations:

$$(\Box+m^2)\phi^{\mu\nu}(x)=0 \tag{3.1}$$

$$\phi^{\mu\nu}(x)-\phi^{\nu\mu}(x)=0 \tag{3.2}$$

$$\phi^{\mu}_{\mu}(x)=0 \tag{3.3}$$

$$\partial_\mu \phi^{\mu\nu}(x)=0. \tag{3.4}$$

In the Takahashi–Umezawa formalism all these equations can be combined into one matrix equation, which in its most general form (Bhargawa and Watanabe 1966) is given by

$$\Lambda_{\mu\nu\rho\sigma}(\partial)\phi^{\rho\sigma}(x)=0, \tag{3.5a}$$

where

$$\Lambda_{\mu\nu\rho\sigma}(\partial)=\tfrac{1}{2}(\Box+m^2)(g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho})-\tfrac{1}{2}(g_{\mu\rho}\partial_\nu\partial_\sigma+g_{\mu\sigma}\partial_\nu\partial_\rho+g_{\nu\rho}\partial_\mu\partial_\sigma+g_{\nu\sigma}\partial_\mu\partial_\rho)$$
$$-\alpha(g_{\mu\nu}\partial_\rho\partial_\sigma+g_{\rho\sigma}\partial_\mu\partial_\nu)-\beta\Box g_{\mu\nu}g_{\rho\sigma}-\gamma m^2 g_{\mu\nu}g_{\rho\sigma}-\delta m^2(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}), \tag{3.5b}$$

and $\beta=\frac{1}{2}(3\alpha^2+2\alpha+1)$, $\gamma=\alpha+2\beta$ and $\delta\neq 0$ are real numbers. The Lagrangian density is

$$\mathscr{L}=-\phi^*_{\mu\nu}(x)g^{\mu\rho}g^{\nu\sigma}\Lambda_{\rho\sigma\kappa\tau}(\partial)\phi^{\kappa\tau}(x). \tag{3.6}$$

Notice that $\mathscr{L}$ is not unique; it contains two real parameters α and δ. Equation (3.5) can be reduced to equations (3.1)–(3.4) by a finite number of differentiations and algebraic operations. This can be carried out by using Klein–Gordon divisor

$$\begin{aligned} d^{\mu\nu\rho\sigma}(\partial) &= \tfrac{1}{2}(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})-\tfrac{1}{3}g^{\mu\nu}g^{\rho\sigma} \\ &\quad -\frac{1}{3m^2}(g^{\mu\nu}\partial^\rho\partial^\sigma+g^{\rho\sigma}\partial^\mu\partial^\nu)+\frac{1}{2m^2}(g^{\mu\rho}\partial^\nu\partial^\sigma+g^{\mu\sigma}\partial^\nu\partial^\rho+g^{\nu\rho}\partial^\mu\partial^\sigma+g^{\nu\sigma}\partial^\mu\partial^\rho) \\ &\quad -\frac{\Box+m^2}{m^2}\left[-\frac{\alpha}{3}\frac{(\alpha+1)}{(2\alpha+1)^2}g^{\mu\nu}g^{\rho\sigma}+\frac{(\alpha+1)^2}{6(2\alpha+1)^2}\frac{\Box}{m^2}g^{\mu\nu}g^{\rho\sigma}\right. \\ &\quad \left. +\frac{1}{3}\frac{(\alpha+1)}{(2\alpha+1)m^2}(\partial^\mu\partial^\nu g^{\rho\sigma}+g^{\mu\nu}\partial^\rho\partial^\sigma)+\frac{1}{\delta}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\right] \\ &\equiv d_1^{\mu\nu\rho\sigma}(\partial)+d_2^{\mu\nu\rho\sigma}(\Box+m^2) \end{aligned} \tag{3.7}$$

$d_2^{\mu\nu\rho\sigma}(\partial)(\Box+m^2)$ are contact terms and are parameter-dependent; they can be made to vanish by a suitable choice of parameters with $\alpha=-1$ and $\delta=0$. The two-point Wightman function does not depend on the contact term and is given by

$$W^{\mu\nu\rho\sigma}(x-y)=d_1^{\mu\nu\rho\sigma}(\partial)W(x-y), \tag{3.8}$$

which is positive semi-definite and one can construct the relativistic one-particle space and Fock space in the usual manner.

To get the correct two-point Schwinger function, one should apply a slightly different matrix transformation on $W^{\mu\nu\rho\sigma}(x-y)$. For if we apply the same matrix transformation as before, then

$$\begin{aligned} S'_{ijmn}(x-y) &= A_{i\mu}A_{j\nu}A_{m\rho}A_{n\sigma}W^{\mu\nu\rho\sigma}(\boldsymbol{x}-\boldsymbol{y}, \mathrm{i}(x_0-y_0)) \\ &= \tfrac{1}{2}(d_{im}d_{jn}+d_{in}d_{jm}-\tfrac{2}{3}d_{ij}d_{mn})S(x-y) \end{aligned}$$

where

$$d_{ij}=\delta_{ij}-m^{-2}\partial_i\partial_j.$$

S'_{ijmn} is not positive semi-definite since it contains a term $-\frac{1}{12}\delta_{ij}\delta_{mn}$ associated with the trace (S'_{ijmn} is not traceless), which contributes a negative norm to the Schwinger function. In order to obtain a positive semi-definite Schwinger function, we must make S'_{ijmn} traceless. This can be achieved by the following matrix transformation:

$$S_{ijmn}(x-y)=(A_{i\mu}A_{j\nu}+\tfrac{1}{4}\delta_{ij}g_{\mu\nu})(A_{m\rho}A_{n\sigma}+\tfrac{1}{4}\delta_{mn}g_{\rho\sigma})W^{\mu\nu\rho\sigma}(\boldsymbol{x}-\boldsymbol{y}, \mathrm{i}(x_0-y_0)). \tag{3.9}$$

Since the relativistic tensor field is traceless (equation (3.3)), the extra term $\frac{1}{4}\delta_{ij}g_{\mu\nu}$ contributes only ultralocal terms to the Schwinger function, consisting of a δ function and its derivatives. It can be shown by direct computation that $\Sigma_i S_{iimn}=0$ and S_{ijmn} is positive semi-definite.

The Euclidean one-particle space $\mathscr{H}$ can be taken as the completion of the symmetric tensor test-function space $\mathscr{S}^{10}(\mathbb{R}^4)$ with respect to the inner product

$$\langle f, g\rangle_{\mathscr{H}}=\sum_{\substack{i<j\\ m<n}}\langle f_{ij}, S_{ijmn}g_{mn}\rangle_{\mathscr{L}^2(\mathbb{R}^4)}. \tag{3.10}$$

The Euclidean massive spin-2 field Φ is defined as the generalized random tensor Gaussian field over $\mathscr{H}$ with mean zero and covariance $E[\Phi(f)\Phi(g)]=\langle f, g\rangle_{\mathscr{H}}$. The Euclidean tensor field Φ is not divergenceless; just like the Euclidean Proca field, $\Sigma_i\,\partial_i\Phi_i$ is ultralocal. The analogues of theorems 1–4 hold for Φ; the proofs are similar to that for the scalar field so we shall omit them. Note that $A=(A_{i\mu}A_{j\nu}+\frac{1}{4}\delta_{ij}g_{\mu\nu})(A_{m\rho}A_{n\sigma}+\frac{1}{4}\delta_{mn}g_{\rho\sigma})$ is a non-singular matrix, hence it can be inverted. Then the Fourier transform of the Schwinger function, $\tilde{S}(p)=A\,W(\boldsymbol{p}, ip_0)$ has a local inverse $\tilde{S}^{-1}(p)=W^{-1}(\boldsymbol{p}, ip_0)A^{-1}=\Lambda(\boldsymbol{p}, ip_0)A^{-1}$, which ensures the Markovicity of Φ.

4. Conclusions

We have constructed Euclidean Markov fields with spin $S\leq 2$, starting from the local Lagrangian formulation of Takahashi and Umezawa. The main advantage of this method is that the Markov property follows immediately from the existence of a local Lagrangian density. It appears that all the algebraic properties (such as symmetry and tracelessness) of the relativistic fields need to be preserved in the Euclidean region. However, the differential condition, i.e. divergenceless of the relativistic fields, is not preserved. In fact, the divergences of Euclidean vector and tensor fields are all ultralocal. It is interesting to note that a field which requires the introduction of indefinite metric Hilbert space in the Minkowski region can have a perfectly proper Euclidean theory with a positive metric Hilbert space, except that the reflection property is violated. This implies that there is a close connection between the indefiniteness of the Hilbert space for a relativistic field and the violation of the reflection property for the corresponding Euclidean Markov field. We shall give more examples to illustrate this point in our next paper which deals with massless fields.

The possibility of extending the above method to Euclidean tensor fields with spin $S\geq 3$ is still under investigation. We expect to face certain complications in such a generalization because the arbitrary parameters present in the local Lagrangian density increase as the value of spin increases, and it would be difficult to get a consistent set of parameters. In our preliminary study there is an indication that for a spin-3 tensor field the Umezawa–Takahashi formulation does not work and auxiliary fields need to be introduced. However, this does not imply that Euclidean spin-3 tensor field is not Markovian, except that the proof of Markovicity may be more involved. We hope to discuss these points in a future paper.

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